

Problem 5:

Given conditional P.M.F of X ,

$$P(X_1 = 1/Q) = Q$$

$$P(X_1 = 0/Q) = 1-Q$$

$$E(X_1|Q) = Q(1) + (1-Q)(0) = Q$$

Also $Q \sim U(0,1)$ and $N \sim N(2,1)$. Then $E(Q) = 0.5$,

$$f_Q(Q) = 1; 0 < Q < 1$$

Define $Y_1 = X_1 + N$, $Y_1|Q \sim N(Q+2, 1)$

Conditional Expectation is

$$E(Y_1|Q) = E((X_1 + N)|Q)$$

$$E(Y_1|Q) = E(X_1|Q) + E(N|Q)$$

$$E(Y_1|Q) = Q + 2$$

$$E(Y_1) = E[E(Y_1|Q)]$$

$$E(Y_1) = E(Q+2)$$

$$E(Y_1) = E(Q) + 2$$

$$E(Y_1) = 0.5 + 2$$

$$E(Y_1) = 2.5$$

$$\Rightarrow E(Y_1) = 2.5$$

Variance

$$\text{Var}(Y_1|Q) = E[(Y_1 - E(Y_1|Q))^2|Q]$$

$$\text{Var}(Y_1|Q) = E[Y_1^2 - 5Y_1 + 6.25|Q]$$

$$\text{Var}(Y_1|Q) = E[Y_1^2|Q] - 5E[Y_1|Q] + E[6.25|Q]$$

$$\text{Var}(Y_1|Q) = 1 + (Q+2)^2 - 5(Q+2) + 6.25$$

$$\text{Var}(Y_1) = E[\text{Var}(Y_1|Q)]$$

$$= \text{Var}(Y_1) = 5 + 4E(Q) + E(Q)^2 - 5E(Y_1) + 6.25$$

$$\text{Var}(Y_1) = 5 + 4 \times 0.5 + 1/4 + 0.5^2 - 5 \times 2.5 + 6.25$$

$$\text{Var}(Y_1) = 13/4$$

Conditional Probability.

$$P(A|Q) = (1-q)^2$$

The unconditional likelihood is.

$$P(A) = \int_0^1 (1-q)^2 f_Q(q) dq$$

$$P(A) = \int_0^1 (1-q)^2 dq$$

$$P(A) = \frac{1}{3}$$

The conditional probability.

$$f_{Q|A}(q|A) = \frac{P(A|Q) f_Q(q)}{P(A)}$$

$$f_{Q|A}(q|A) = \frac{(1-q)^2}{\frac{1}{3}}$$

$$f_{Q|A}(q|A) = \underline{\underline{3(1-q)^2; 0 < q < 1}}$$